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THERMOELASTIC STRESSES IN A PLANE WITH A CIRCULAR INCLUSION IN THE PRESENCE OF A THERMAL SPOT OF ELLIPTICAL SHAPE

L. G. Smirnov and I. I. Fedik UDC 539.412

The problem of determining the thermostress state of a body during heating by a spot occupying a certain domain reduces to a problem of determining the elastic stresses for given discontinuities in the displacements on the spot boundary [1]. This latter is equivalent to the problem of determining the elastic stresses caused by the presence of an inclusion preliminarily subjected to intrinsic strain and having elastic characteristics as also the surrounding medium and then inserted in the hole occupied by the spot domain [2]. Utilization of the Muskhelishvili method in the plane case permits reducing this problem to a standard boundary value problem of elasticity theory for the whole domain occupied by the body with altered external forces [2]. When the spot is circular in shape, the solution can be found in closed form [3-5]. The solution of the problem of determining the stresses in a half-plane for an elliptical spot shape and constant magnitude of the heating ΔT is also written in closed form [6]. This paper is devoted to obtaining such a solution for a plane with a circular foreign inclusion for an elliptical spot shape and ΔT = const.

Let an elastic plane with a circular foreign inclusion be heated over a certain domain D bounded by the contour L from an initial temperature T_0 for which there is no stress state to a temperature T_1. It is assumed that the contour L does not intersect the circle L_0 bounding the foreign inclusion and can be a system of nonintersecting closed contours L_j (j = 1, 2, ..., n). Without limiting the generality, we will consider the contour L to consist of two contours L_1 and L_2 bounding domains D_1^+ and D_2^+ lying entirely within and outside the circle L_0, respectively. The domain lying between the contours L_0 and L_1 is denoted by D_1^- and the domain between L_0 and L_2 by D_2^-. It is known [2] that the stress state that occurs is equivalent to that which occurs in inclusions occupying the domain D_j^+ first subjected to intrinsic strain and from the same material as its external medium, and then installed in holes with the contours L_j (j = 1, 2, ..., n).

Let us assume the center of the circular foreign inclusion of radius R_0 to be at the origin of the x, y plane, and μ_j, ν_j, α_j to be the shear modulus, Poisson ratio, and coefficient of thermal expansion of the materials of the foreign inclusion (j = 1) and its external medium (j = 2). We use the Muskhelishvili method to find the stress state. Considering that an ideal contact holds on the common boundary of the inclusion with the medium, the conditions of equality of the normal and tangential stresses as well as the presence of a displacement jump on the interfacial lines of the media caused by the intrinsic strains are written in the form

$$\varphi_0^-(t) + t\overline{\varphi_0^{-\prime}(t)} + \overline{\psi_0^-(t)} = \varphi^-(t) + t\overline{\varphi^{-\prime}(t)} + \overline{\psi^-(t)} + C_1, \quad (1)$$
$$\left(\varkappa_1\varphi_0^-(t) - t\overline{\varphi_0^{-\prime}(t)} - \overline{\psi_0^-(t)}\right)/\mu_1 = \left(\varkappa_2\varphi^-(t) - t\overline{\varphi^{-\prime}(t)} - \overline{\psi^-(t)}\right)/\mu_2 \quad (t\in L_0);$$

$$\varphi_0^+(t) + t\overline{\varphi_0^{+\prime}(t)} + \overline{\psi_0^+(t)} = \varphi_0^-(t) + t\overline{\varphi_0^{-\prime}(t)} + \overline{\psi_0^-(t)} + C_2, \quad (2)$$
$$\varkappa_1\varphi_0^+(t) - t\overline{\varphi_0^{+\prime}(t)} - \overline{\psi_0^+(t)} = \varkappa_2\varphi_0^-(t) - t\overline{\varphi_0^{-\prime}(t)} - \overline{\psi_0^-(t)} + 2\mu_1 g_1(t) \quad (t\in L_1);$$

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$$\varphi^+(t) + t\overline{\varphi^{+\prime}(t)} + \overline{\psi^+(t)} = \varphi^-(t) + t\overline{\varphi^{-\prime}(t)} + \overline{\psi^-(t)} + C_3,$$
$$\varkappa_1\varphi^+(t) - t\overline{\varphi^{+\prime}(t)} - \overline{\psi^+(t)} = \varkappa_2\varphi^-(t) - t\overline{\varphi^{-\prime}(t)} - \overline{\psi^-(t)} + 2\mu_2 g_2(t) \quad (t \in L_2). \tag{3}$$

Here $\varphi_0^+(t)$, $\psi_0^+(t)$ are the boundary values of functions holomorphic in the domain D_1^+, $\varphi_0^-(t)$, $\psi_0^-(t)$ are boundary values of functions holomorphic in the domain D_1^-, $\varphi^+(t)$, $\psi^+(t)$ in the domain D_2^+, $\varphi^-(t)$, $\psi^-(t)$ in the domain D_2^- $(t = x + iy)$; $\varkappa_j = (3 - \nu_j)/(1 + \nu_j)$ and $\varkappa_j = 3 - 4\nu_j$ in the cases of the plane stress and plane strain states, $g_j(t) = u_j + iv_j$ are displacement jumps on the lines L_j $(j = 1, 2)$. It is easily seen that the constants C_j $(j = 1, 2, 3)$ can be included in the desired functions and consequently, we shall set $C_j = 0$ $(j = 1, 2, 3)$ [2].

As is shown in [2], by using conditions (2) and (3), the functions $\varphi_0(z)$, $\psi_0(z)$ and $\varphi(z)$, $\psi(z)$, equal to $\varphi_0^\pm(z)$, $\psi_0^\pm(z)$ for $z \in D_1^\pm$ and $\varphi^\pm(z)$, $\psi^\pm(z)$ for $z \in D_2^\pm$ can be represented as

$$\varphi_0(z) = \varphi_1(z) + \varphi_{1*}(z), \qquad \psi_0(z) = \psi_1(z) + \psi_{1*}(z);$$
$$\varphi(z) = \varphi_2(z) + \varphi_{2*}(z), \qquad \psi(z) = \psi_2(z) + \psi_{2*}(z), \tag{4}$$

where the functions $\varphi_1(z)$, $\psi_1(z)$ and $\varphi_2(z)$, $\psi_2(z)$ are holomorphic, respectively, inside and outside L_0 and the following representation holds

$$\varphi_{j*}(z) = \frac{\mu_j}{\pi i(1+\varkappa_j)} \int_{L_j} \frac{g_j(t)\,dt}{t-z},$$
$$\psi_{j*}(z) = \frac{\mu_j}{\pi i(1+\varkappa_j)} \int_{L_j} \frac{h_j(t)\,dt}{t-z}, \tag{5}$$

where

$$h_j(t) = -\overline{g_j(t)} - \bar{t}g_j'(t) \quad (j = 1, 2). \tag{6}$$

Substituting (4) into conditions (1), we obtain

$$\varphi_1(t) + t\overline{\varphi_1'(t)} + \overline{\psi_1(t)} = \varphi_2(t) + t\overline{\varphi_2'(t)} + \overline{\psi_2(t)} + f_1(t); \tag{7}$$
$$\varkappa_1\varphi_1(t) - t\overline{\varphi_1'(t)} - \overline{\psi_1(t)} = \gamma(\varkappa_2\varphi_2(t) - t\overline{\varphi_2'(t)} - \overline{\psi_2(t)}) + f_2(t), \tag{8}$$

where

$$\gamma = \mu_1/\mu_2,\ f_1(t) = p_2(t) - p_1(t),\ f_2(t) = \gamma q_2(t) - q_1(t),$$
$$p_j(t) = \varphi_{j*}(t) + t\overline{\varphi_{j*}'(t)} + \overline{\psi_{j*}(t)},\quad q_j(t) = \varkappa_j\varphi_{j*}(t) - t\overline{\varphi_{j*}'(t)} - \overline{\psi_{j*}(t)}$$
$$(j = 1, 2). \tag{9}$$

The problem therefore reduces to finding the holomorphic functions $\varphi_j(z)$, $\psi_j(z)$ $(j = 1, 2)$ inside $(j = 1)$ and outside $(j = 2)$ the circle L_0 $(|t| = R_0)$ according to the conjugate conditions (7) and (8). Taking into account that the equality $\bar{t} = R_0^2/t$ holds for $t \in L_0$, it is easily seen that the functions $\overline{\varphi_1(t)}$, $\overline{\psi_1(t)}$ and $\overline{\varphi_2(t)}$, $\overline{\psi_2(t)}$ are the boundary values of functions holomorphic, respectively, inside and outside the contour L_0. Indeed, since $\varphi_1(z)$, $\psi_1(z)$ are holomorphic inside L_0 while $\varphi_2(z)$, $\psi_2(z)$ are holomorphic outside L_0, then $\varphi_j(z)$ and $\psi_j(z)$ $(j = 1, 2)$ are represented in the form

$$\varphi_1(z) = \sum_0^\infty a_n z^n,\quad \psi_1(z) = \sum_0^\infty b_n z^n.\quad \varphi_2(z) = \sum_0^\infty c_n z^{-n},\quad \psi_2(z) = \sum_0^\infty d_n z^{-n}, \tag{10}$$

meaning

$$\overline{\varphi_1'(t)} = \sum_0^\infty n\overline{a_n t^{n-1}} = \sum_0^\infty n\bar{a}_n R_0^{2n-2} t^{-n+1},\quad \overline{\psi_1(t)} = \sum_0^\infty \bar{b}_n R_0^{2n} t^{-n},$$

$$\overline{\varphi_2'(t)} = -\sum_0^\infty n\bar{c}_n \overline{t^{-n-1}} = -\sum_0^\infty n\bar{c}_n R_0^{-2n-2} t^{n+1}, \quad \overline{\psi_2(t)} = \sum_0^\infty \bar{d}_n R_0^{-2n} t^n.$$

Hence, it follows that $\overline{\varphi_1'(t)}$, $\overline{\psi_1(t)}$ and $\overline{\varphi_2'(t)}$, $\overline{\psi_2(t)}$ are boundary values of the functions $\bar{\varphi}_1'(R_0^2/z)$, $\bar{\psi}_1(R_0^2/z)$ and $\bar{\varphi}_2'(R_0^2/z)$, $\bar{\psi}_2(R_0^2/z)$, that are holomorphic outside and inside L_0 ($\bar{F}(z) = \overline{F(\bar{z})}$, differentiation is with respect to the variable $\xi = R_0^2/z$). Multiplying both sides of (7) and (8) by the coefficient $1/(2\pi i(t - z))$ and integrating along L_0 with the above-mentioned from (9) and (10) taken into account, we obtain for the z lying within the circle

$$\varphi_1(z) + \bar{a}_1 z + 2\bar{a}_2 = c_0 + z\bar{\varphi}_2'(R_0^2/z) + \bar{\psi}_2(R_0^2/z) + f_1^+(z); \tag{11}$$

$$\varkappa_1\varphi_1(z) - \bar{a}_1 z - 2\bar{a}_2 = \gamma\left(\varkappa_2 c_0 - z\bar{\varphi}_2'(R_0^2/z) - \bar{\psi}_2(R_0^2/z)\right) + f_2^+(z), \tag{12}$$

where

$$f_j^+(z) = \frac{1}{2\pi i}\int_{L_j} \frac{f_j(t)}{t - z}\,dt. \tag{13}$$

We have from (11) and (12)

$$\varphi_1(z) = \left[\left(f_2^+(z) + \gamma f_1^+(z)\right) + \gamma(1 + \varkappa_2)c_0 + (1 - \gamma)\left(\bar{a}_1 z + 2\bar{a}_2\right)\right]/(\varkappa_1 + \gamma). \tag{14}$$

Analogously we write for z lying outside the circle

$$z\bar{\varphi}_1'(R_0^2/z) + \bar{\psi}_1(R_0^2/z) = \varphi_2(z) - c_0 + f_1^-(z); \tag{15}$$

$$-z\bar{\varphi}_1'(R_0^2/z) - \bar{\psi}_1(R_0^2/z) = \gamma(\varkappa_2\varphi_2(z) - \varkappa_2 c_0) + f_2^-(z). \tag{16}$$

Then

$$\varphi_2(z) = -\left(f_1^-(z) + f_2^-(z) - (1 + \varkappa_2)c_0\right)/(1 + \gamma\varkappa_2) \tag{17}$$

[$f_j^-(z)$ is understood to be the integral in the right side of (13) for z outside the circle];

$$\psi_1(z) = -R_0^2\left(\varphi_1'(z) - a_1\right)/z + \bar{\varphi}_2(R_0^2/z) - \bar{a}_0 + \bar{f}_1^+(R_0^2/z); \tag{18}$$

$$\psi_2(z) = -R_0^2\left(\varphi_2'(z) - a_1\right)/z + \bar{\varphi}_1(R_0^2/z) - \bar{a}_0 - \bar{f}_1^-(R_0^2/z). \tag{19}$$

The constant c_0 in (14)-(17) can be set equal to zero since the problem with conditions (7) and (8) conserves the arbitrariness in determining $\varphi_2(z)$ [or $\psi_2(z)$] to the accuracy of an additive constant. Differentiating both sides of (14) twice and z = 0 we find

$$a_0 = F''(0)/(2(\varkappa_1 + \gamma)) \quad \left(F(z) = f_2^+(z) + \gamma f_1^+(z)\right).$$

For a single differentiation of both sides of (14) at z = 0 we obtain an equation to find a_1

$$a_1 = (F'(0) + (1 - \gamma)(\varkappa_1 + \gamma)^{-1}\overline{F'(0)})/(1 + (\gamma - 1)^2(\varkappa_1 + \gamma)^{-2}). \tag{20}$$

For z = 0 we have $a_2 = a_0(\varkappa_1 + \gamma) - F(0)/(2(1 - \gamma))$ from (14).

Now, let us examine the particular case when the contour L_j is an ellipse with semi-axes a_j^0, b_j^0 and angle θ_j between the directions of the semiaxes a_j^0 and the x axis, while $T_1^{(j)} - T_0^{(j)} = \Delta T^{(j)}$ = const. It is easy to see that

$$g_j(t) = -\varepsilon_j(t - z_j), \quad h(t) = -\varepsilon_j(\bar{z}_j - 2\bar{t}), \quad \varepsilon_j = \alpha_j k_j \Delta T^{(j)}, \tag{21}$$

where z_j is the center of an ellipse with contour L_j, $k_j = 1$ and $k_j = 1 + \nu_j/1 - \nu_j$ in the plane stress and plane strain states cases, respectively. Taking account of (6) and (21) we find the expressions for $\varphi_{1*}(z)$ and $\psi_{1*}(z)$ that are needed to calculate the stresses from (5):

$$\varphi_{j*}(z)=\frac{\mu_j}{\pi i(1+\varkappa_j)}\int_{L_0}\frac{g_j(t)\,dt}{t-z}=\begin{cases}-\beta_j(z-z_j) & (z\in D_j^+),\\ 0 & (z\in D_j^-);\end{cases} \tag{22}$$

$$\psi_{j*}(z)=\frac{\mu_j}{\pi i(1+\varkappa_j)}\int_{L_0}\frac{h_j(t)\,dt}{t-z}=\begin{cases}-\beta_j\bar{z}+2\beta_j I(z) & (z\in D_j^+),\\ 2\beta_j I(z) & (z\in D_j^-)\end{cases} \tag{23}$$

$$\left(\beta_j=2\varepsilon_j\mu_j/(1+\varkappa_j),\quad I(z)=\frac{1}{2\pi i}\int_{L_0}\frac{\bar{t}\,dt}{t-z}\right).$$

To evaluate I(z) we use the conformal mapping function for the exterior of the ellipse L_j onto the exterior of the unit circle γ_0 in the ζ plane which has the form

$$z=\omega_j(\zeta)=R_j(\zeta+m_j\zeta^{-1})\,e^{i\theta_j}+z_j\quad\left(R_j=(a_j^0+b_j^0)/2,\quad m=\frac{a_j^0-b_j^0}{a_j^0+b_j^0}\right).$$

Then taking account of the equality $\bar{\sigma}=1/\sigma$ for $\sigma\in\gamma_0$,

$$I(z)=\frac{1}{2\pi i}\int_{L_0}\frac{\bar{t}\,dt}{t-z}=\frac{e^{-2i\theta_j}}{2\pi i}\int_{\gamma_0}\frac{(m_j\sigma^2-\bar{z}_je^{i\theta_j}\sigma/R_j+1)(1-m_j\sigma^{-2})}{(\sigma^2-(z-z_j)\,e^{-i\theta_j}\sigma/R_j+m_j)}\,d\sigma. \tag{24}$$

Since the equation $\sigma^2-(z-z_j)e^{-i\theta_j}\sigma/R_j+m_j=0$ is the entry for the conformal mapping of the z plane with an elliptic hole on both the exterior and interior of the circle $|\zeta|=1$, then its roots σ_1 and σ_2 lie inside and outside the circle $|\zeta|=1$. Using residue theory, we obtain from (24)

$$I(z)=\begin{cases}e^{-2i\theta_j}(m_j-m_j^{-1})\left(z-z_j-\sqrt{(z-z_j)^2-4m_jR_j^2e^{2i\theta_j}}\right)/2 & (z\in D_j^-),\\ e^{-2i\theta_j}m_j(z-z_j) & (z\in D_j^+)\end{cases} \tag{25}$$

$\left(\text{the branch satisfying the condition } \lim_{|z|\to\infty}I(z)=0\right)$ is selected . Taking account of (22) and (23) we find from (9)

$$f_1(t)=\psi_{2*}(t)-\psi_{1*}(t),\quad f_2(t)=\psi_{1*}(t)-\gamma\psi_{2*}(t). \tag{26}$$

Since $\psi_{1*}(z)$ and $\psi_{2*}(z)$ are, respectively, holomorphic in the domains $D_2^+ + D_2^-$ and $D_1^+ + D_1^-$, the functions $\overline{\psi_{1*}(t)}$ and $\overline{\psi_{2*}(t)}$ are boundary values of the functions $\bar{\psi}_{1*}(R_0^2/z)$ and $\bar{\psi}_{2*}(R_0^2/z)$ that are holomorphic in the domains $D_1^+ + D_1^-$ and $D_2^+ + D_2^-$. Utilizing (13) and (26) we have

$$\begin{aligned}&f_1^-(z)=-\bar{\psi}_{2*}(R_0^2/z),\quad f_2^-(z)=\gamma\bar{\psi}_{2*}(R_0^2/z)\quad(|z|>R_0),\\ &f_1^+(z)=-\bar{\psi}_{1*}(R_0^2/z),\quad f_2^+(z)=\bar{\psi}_{1*}(R_0^2/z)\quad(|z|<R_0).\end{aligned} \tag{27}$$

Differentiating the equalities (14), (17) and (18), (19), with (27) taken into account, we obtain

$$\begin{aligned}&\varphi_1'(z)=(1-\gamma)/(\varkappa_1+\gamma)(\bar{\psi}_{1*}'(R_0^2/z)+\bar{a}_1),\\ &\varphi_1''(z)=(1-\gamma)/(\varkappa_1+\gamma)\,\bar{\psi}_{1*}''(R_0^2/z),\\ &\psi_1'(z)=R_0^2\varphi_1'(z)/z^2-R_0^2\varphi_1''(z)/z+\bar{\varphi}_2'(R_0^2/z)-R_0^2a_1/z^2+\psi_{2*}'(z);\end{aligned} \tag{28}$$

$$\begin{aligned}&\varphi_2'(z)=(1-\gamma)/(1+\gamma\varkappa_2)\,\bar{\psi}_{2*}'(R_0^2/z),\quad \varphi_2''(z)=(1-\gamma)/(1+\gamma\varkappa_2)\,\bar{\psi}_{2*}''(R_0^2/z),\\ &\psi_2'(z)=R_0^2\varphi_2'(z)/z^2-R_0^2\varphi_2''(z)/z+\bar{\varphi}_1'(R_0^2/z)-R_0^2a_1/z^2+\psi_{1*}'(z).\end{aligned} \tag{29}$$

The derivatives $\bar{\psi}_{j*}'(R_0^2/z)$ and $\bar{\psi}_{j*}''(R_0^2/z)$ in (28) and (29) are found from the formulas

$$\overline{\psi}'_{j*}(R_0^2/z) = \overline{d\psi_{j*}(R_0^2/\bar z)/dz} = (\overline{d\psi_{j*}(R_0^2/\bar z)/d\bar z})(\overline{d\bar z/dz}) =$$
$$= \overline{\psi_{j*}(R_0^2/\bar z)'_\eta(-R_0^2/\bar z^2)(d\bar z/dz)}(\overline{d\bar z/dz}) = -R_0^2\overline{\psi_{j*}(R_0^2/\bar z)'_\eta}/z^2, \tag{30}$$
$$\overline{\psi}''_{j*}(R_0^2/z) = 2R_0^2\overline{\psi_{j*}(R_0^2/\bar z)'_\eta}/z^3 + R_0^4\overline{\psi_{j*}(R_0^2/\bar z)''_\eta}/z^4 \quad (\eta = R_0^2/\bar z).$$

It follows from (14) that

$$\overline{\varphi}_1(R_0^2/z) = \overline{\varphi_1(R_0^2/\bar z)} = (1-\gamma)/(\varkappa_1+\gamma)(\psi_{1*}(z) + a_1R_0^2/z),$$

from which

$$\overline{\varphi}'_1(R_0^2/z) = (1-\gamma)/(\varkappa_1+\gamma)(\psi'_{1*}(z) - a_1R_0^2/z^2). \tag{31}$$

We obtain from (17) in an analogous manner

$$\overline{\varphi}'_2(R_0^2/z) = (\gamma-1)/(1+\gamma\varkappa_1)\psi'_{2*}(z).$$

The coefficient a_1 is found from (20), where $F(z) = (1-\gamma)\bar\psi_{j*}(R_0^2/z)$. Taking account of (23) and (25) the functions $\psi_{j*}'(z)$, $\bar\psi_{j*}'(R_0^2/z)$, $\bar\psi_{j*}''(R_0^2/z)$ have the form

$$\psi'_{j*}(z) = \begin{cases}\delta_j\beta_j[1-(z-z_j)/((z-z_j)^2-\gamma_j^2)^{1/2}] & (z\in D_j^-),\\ \lambda_j\beta_j & (z\in D_j^+);\end{cases} \tag{32}$$

$$\overline{\psi}'_{j*}(R_0^2/z) = -\bar\delta_j\beta_jR_0^2[1-(R_0^2/z-\bar z_j)/(\overline{(R_0^2/\bar z-z_j)^2-\gamma_j^2})^{1/2}]z^{-2}; \tag{33}$$

$$\overline{\psi}''_{j*}(R_0^2/z) = \bar\delta_j\beta_j[2R_0^2(1-(R_0^2/z-\bar z_j)/(\overline{(R_0^2/\bar z-z_j)^2-\gamma_j^2})^{1/2} + R_0^4\bar\gamma_j^2/(\overline{(R_0^2/\bar z-z_j)^2-\gamma_j^2})^{3/2}z^{-1})]z^{-3}. \tag{34}$$

Here $\delta_j = (m_j - m_j^{-1})e^{-2i\theta_j}$; $\gamma_j^2 = 4m_jR_j^2e^{2i\theta_j}$; $\lambda_j = 2m_je^{-2i\theta_j}$ while $|z| < R_0$ for $j = 1$, $|z| > R_0$ for $j = 2$. In the case when L_j is a circle, by passing to the limit as $m_j \to 0$, we have from (32)-(34)

$$\psi'_{j*}(z) = \begin{cases}2\beta_jR_j^2(z-z_j)^{-2} & (z\in D_j^-),\\ 0 & (z\in D_j^+),\end{cases}$$
$$\overline{\psi}'_{j*}(R_0^2/z) = 2\beta_jR_0^2R_j^2(R_0^2-z\bar z_j)^{-2},\quad \overline{\psi}''_{j*}(R_0^2/z) = 4\beta_jR_0^2R_j^2\bar z_j(R_0^2-z\bar z_j)^{-3}$$
$$(j = 1,\ 2).$$

The desired stresses are found from the formulas [2]

$$\sigma_{xx} + \sigma_{yy} = 4\mathrm{Re}(\Phi_j(z)),$$
$$\sigma_{yy} - \sigma_{xx} + 2i\tau_{xy} = 2(\bar z\Phi_j(z) + \Psi_j(z)) \quad (j = 1,\ 2); \tag{35}$$
$$\Phi_j(z) = \varphi'_{j*}(z) + \varphi'_j(z),\quad \Psi_j(z) = \psi'_{j*}(z) + \psi'_j(z) \quad (j = 1,\ 2).$$

Taking account of (30), (31), and (35), it is easy to obtain a solution from (28) and (29) for the case when the circle degenerates into the line $z = 0$. Indeed, the functions $\Phi_j(z)$, $\Psi_j(z)$ are converted upon substitution of the coordinates $z = z' - R_0$ according to the formulas [2]

$$\Phi_j(z) = \Phi(z'-R_0),\quad \Psi_j(z) = \Psi_j(z'-R_0) - R_0\Phi'_j(z'-R_0).$$

Let us use the notation $\Phi_j(z) = \tilde\Phi_j(z')$ and $\Psi_j(z) = \tilde\Psi_j(z')$:

$$\tilde\Phi_j(z') = \Phi_j(z'-R_0) = l_j\overline{\psi}'_{j*}(R_0^2/(z'-R_0)) + \varphi_{j*}(z'-R_0) +$$
$$+ l_j(2-j)\bar a_1 = -l_j\beta_j\bar\delta_jR_0^2/(R_0-z')^2\times \tag{36}$$
$$\times[1-(R_0^2/(z'-R_0)-(\bar z_j-R_0))/(\overline{(R_0^2/(\bar z'-R_0)-z'+R_0)^2-\gamma_j^2})^{1/2}] +$$
$$+\varphi'_{j*}(z'-R_0) + l_j(2-j)\bar a_1 \quad (j = 1,\ 2).$$

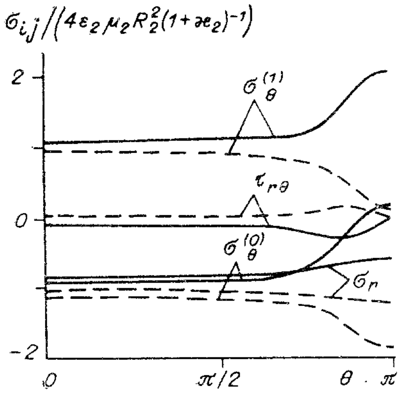

Fig. 1

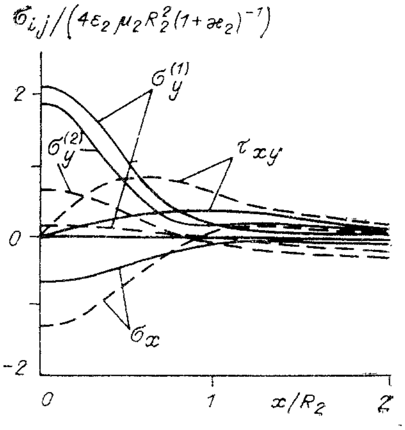

Fig. 2

Here

$$l_1=(1-\gamma)/(\varkappa_1+\gamma),\ l_2=(\gamma-1)/(1+\gamma\varkappa_2),$$

$$\varphi'_{j*}(z'-R_0)=\varphi'_{j*}(z)=\begin{cases}-\beta_j & (z\in D_j^+),\\ 0 & (z\in D_j^-)\end{cases}\quad (j=1,\ 2).$$

Since $F(z)=(1-\gamma)\bar{\psi}_{1*}(R_0^2/(z'-R_0))$, then by using (20), (23), and (25) we write $\lim\limits_{R_0\to\infty}F(z)=\widetilde{F}(z')=0$, which means $a_1=0$. Passing to the limit in (36), we obtain

$$\widetilde{\Phi}_j(z')=\lim_{R_0\to\infty}\Phi_j(z'-R_0)=-l_j\beta_j\bar{\delta}_j\left(1-(z'-\bar{z}_j')/\overline{((\bar{z}'-z_j')^2-\gamma_j^2)^{1/2}}\right)+\psi'_{j*}(z'),$$

$$\widetilde{\Phi}'_j(z')=\gamma_j^{-2}l_j\beta_j\bar{\delta}_j/\overline{((\bar{z}'-z_j')^2-\gamma_j^2)^{3/2}}+\varphi''_{j*}(z')\quad (j=1,\ 2).$$

Analogously

$$\widetilde{\Psi}_j(z')=\lim_{R_0\to\infty}\{\Psi_j(z'-R_0)-R_0\Phi'_j(z'-R_0)\}=$$

$$=\lim_{R_0\to\infty}\{R_0^2(z'-R_0)^{-2}\varphi'_j(z'-R_0)-R_0^2(z'-R_0)^{-1}\varphi''_j(z'-R_0)+$$

$$+l_{(3-j)}\psi'_{(3-j)*}(z'-R_0)+\psi'_{j*}(z'-R_0)+\psi'_{(3-j)*}(z'-R_0)-R_0\varphi''_j(z-R_0)\}=$$

$$=\widetilde{\Phi}_j(z')+(l_{(3-j)}+1)\widetilde{\psi}'_{(3-j)}(z')+\widetilde{\psi}'_{j*}(z')\quad (j=1,\ 2).$$

The domains $x>0$ and $x<0$ correspond to the domains $D_1^+ + D_1^-$ and $D_2^+ + D_2^-$. When L_j is a circle we find by returning to the old notation ($z\to z'$, $(\Phi,\Psi)\to(\tilde{\Phi},\tilde{\Psi})$):

$$\Phi_j(z)=b_j\left((z-\bar{z}_j)^{-2}-d_j/2\right),\quad \Phi'_j(z)=-2b_j(z-\bar{z}_j)^{-3},$$

$$\Psi_j(z)=b_j(\bar{z}_j(z-\bar{z}_j)^{-3}-(z-\bar{z}_j)^{-2}/2)+$$

$$+(l_{(3-j)}+1)c_{(3-j)}(z-\bar{z}_j)^{-2}+c_jd_j(z-z_j)^{-2},$$

$$c_j=4\varepsilon_j\mu_jR_0^2/(1+\varkappa_j),\quad b_j=l_jc_j,\quad d_j=\begin{cases}1 & (z\in D_j^+),\\ 0 & (z\in D_j^-)\end{cases}\quad (j=1,\ 2).$$

Graphs of the stresses are represented in Figs. 1 and 2 in dimensionless form in conformity with the formula $\sigma_{k\ell}^{-(j)}=\sigma_{k\ell}^{(j)}/(4\varepsilon_2\mu_2R_2^2(1+\varkappa_2)^{-1})$ on the outer and inner boundaries of the contours L_2 and L_0 in the case when the contour L_2 is the circle $z-z_2=R_2e^{i\theta}$, the contour L_0 is the line $x=0$ and $\varepsilon_1=0$. The stresses σ_{rr}, $\sigma_{\theta\theta}$, and $\tau_{r\theta}$ (see Fig. 1) are computed from the formulas

$$\sigma_{rr}+\sigma_{\theta\theta}=4\mathrm{Re}\,(\varphi'(z)),$$

$$\sigma_{\theta\theta}-\sigma_{rr}+2i\tau_{r\theta}=2e^{2i\alpha}\,\overline{(\bar{z}\varphi''(z)+\psi'(z))}$$

($e^{2i\alpha} = z^2r^{-2}$) and σ_{xx}, σ_{yy}, τ_{xy} (see Fig. 2) according to the same formulas for $\alpha = 0$. The computations were performed for $\varkappa_1 = \varkappa_2 = 2$; $\gamma = 3^{-1}$; 3; $z_2 = 1.001R_2$. The solid lines in Figs. 1 and 2 are $\gamma = 1/3$, the dashes are $\gamma = 3$, and the superscripts j = 0, 1, 2 correspond to the domains D_2^+, D_2^-, D_1^-.

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VIBRATIONS OF AN ELASTIC ORTHOTROPIC LAYER WITH A CAVITY

A. O. Vatul'yan and A. Ya. Katsevich

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In connection with the development of vibrational seismographic prospecting and defectometry at the present time, problems of analyzing wave fields in an elastic medium with cavities, cracks, and inclusions became extremely urgent. Let us note that certain materials being tested are anisotropic (austenite class steels, composites, soils) which requires an appropriate mathematical model that takes account of the anisotropy of the mechanical properties.

1. The steady-state antiplane waves are investigated in an orthotropic elastic layer of thickness h with a cylindrical cavity whose directrix is a smooth closed curve ℓ_0. We consider that the vibrations in the layer are excited by a tangential load $p(x_1)$ applied to the boundary $x_3 = h$ of the layer. The axes of elastic symmetry agree with the coordinate axes, the component $u_2 = u(x_1, x_3)\exp(-i\omega t)$ of the displacement vector components is different from zero while similarly $\sigma_{12} = c_{66}u_{,1}$, $\sigma_{23} = c_{44}u_{,3}$ from the stress tensor components. After extraction of the time factor the boundary value problem has the form

$$\begin{gathered} c_{66}u_{,11} + c_{44}u_{,33} + \rho\omega^2 u = 0, \\ x_3 = h,\ c_{44}u_{,3} = p(x_1),\ x_3 = 0,\ u = 0, \\ (x_1,\ x_3) \in l_0,\ c_{66}u_{,1}n_1 + c_{44}u_{,3}n_3 = 0 \end{gathered} \tag{1.1}$$

(n_1, n_3 are components of the unit vector normal to the curve ℓ_0, external relative to the domain occupied by the elastic medium). Formulation of the problem is closed by the radiation condition for whose formulation the limit absorption principle is used.

We introduce an auxiliary boundary value problem for the function $U(x_1, x_3, \xi_1, \xi_3)$ into the consideration

$$\begin{gathered} c_{66}U_{,11} + c_{44}U_{,33} + \rho\omega^2 U = -\delta(x_1 - \xi_1,\ x_3 - \xi_3), \\ x_3 = h,\ U_{,3} = 0,\ x_3 = 0,\ U = 0. \end{gathered} \tag{1.2}$$

The solution of the problem (1.2) is constructed by using a Fourier integral transform within

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